A PARALLELIZED QUASI-MONTE CARLO ALGORITHM FOR THE EXTRACTION OF PARTIAL INDUCTANCES IN IC INTERCONNECT STRUCTURES

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Abstract: This paper presents a parallelized Quasi-Monte Carlo algorithm for the extraction of partial inductances in IC interconnect structures. Quasi-random numbers provide superior convergence over pseudo-random numbers, and in this paper the results are presented for three quasi-random number sequences-Halton, Sobol and Niederreiter. The algorithm has been parallelized and an almost linear rate of parallelization is achieved.

Keywords: Monte Carlo, Stochastic Algorithm, IC Interconnect Modeling, Inductance Extraction, Partial Inductance

1. Introduction

With operating frequencies in the GHz range, the role of on-chip inductance is becoming increasingly significant in IC design. The inclusion of inductance in the interconnect model is of particular importance in clock distribution networks, signal and power lines, which have wide wires and hence low resistance. The advent of low resistance copper interconnects has further increased the importance of inductance in IC design.

The conventional approaches to inductance extraction involve loop inductance models [1]. In these loop inductance models, typically the capacitive effects are omitted during resistance and inductance extraction. A RLC model is then constructed by adding lumped or distributed interconnect capacitance. A radically different approach [2] to the modeling of inductance has been suggested in literature, which precludes the need to determine the current distribution in advance. This approach is based on the Partial Element Equivalent Circuit (PEEC) [3] method. In this approach, the interconnect lines are divided into wire segments and self and mutual inductances are extracted for these “partial elements”. These extracted inductances are then stitched together with various resistances and capacitances to form an effective RLC circuit model. It has been demonstrated [2] that this PEEC-based approach is more accurate than the loop inductance models in that the latter overestimates the signal delay time and the undershoot. The primary reason behind this lies in the fact the PEEC-based models take into account the mutual inductances between the different “partial elements” of a particular loop, while the loop inductance models take into account only the mutual inductance between different loops. The subject of this paper is the Monte Carlo-based extraction of mutual inductances between these
partial elements through the use of low discrepancy sequences or quasi-random numbers (QRNs) [4-6].

2. Integral Formulation for Inductance and the Monte Carlo Method

The most general formulation [7] for self and mutual inductance in conductor systems follows from a definition of inductance based on magnetic energy. The mutual inductance between two partial elements is given by:

\[
M_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_{v_i} d^3x_i \int_{v_j} d^3x_j \frac{\mathbf{J}(x_i) \cdot \mathbf{J}(x_j)}{|x_i - x_j|}.
\]

Above, the mutual inductance \(M_{ij}\) is formulated as a six-dimensional integral over the position coordinates over the \(i\)-th and the \(j\)-th conductor; \(x, v, J\) with appropriate suffix represent the position coordinate, volume and current density; \(d^3x\) with an appropriate suffix represents an infinitesimally small volume element and \(\mu_0\) is the magnetic permeability of free space. It can be seen that under the assumption of constant current density, the expression for mutual inductance reduces to an integral dependent on position coordinates and easily generalizes for partial elements that can be represented as one- and two-dimensional structures.

We will now describe briefly the fundamentals of the Monte Carlo integration technique used in this work, known as the Sample Mean Monte Carlo [8]. Let us consider a function \(f(x)\) defined over the interval \(a \leq x \leq b\). The objective is to estimate the integral

\[
I = \int_a^b dx f(x).
\]

In the event, the integral is improper, absolute convergence [9] is assumed. We select an arbitrary probability density function \(p(x)\). A random variable \(\xi\) is defined corresponding to a probability density function \(p(x)\). We now introduce another random variable \(\kappa\) defined as

\[
\kappa = \frac{f(\xi)}{p(\xi)}.
\]

The expectation value of the random variable \(\kappa\), written as \(M(\kappa)\), is then an estimate of the integral \(I\), which can be rewritten as

\[
I = M(\kappa) = \int_a^b dx \left[ \frac{f(x)}{p(x)} \right] p(x).
\]

The integral can be evaluated by sampling the quantity \([f(x)/p(x)]\) according to the probability density function \(p(x)\) with the help of a random-number generator [10] and averaging over a statistically large number of such samples. It can be noted that the Monte Carlo integration technique is ideally adapted to the estimation of multi-dimensional integrals such as the one in Eq. (1), as only the integrand needs to be sampled irrespective of the dimensionality of the integral. Furthermore, the integration technique is inherently parallelizable, as the samples of the
integrand are stochastically independent. As a result, the integrand can be easily sampled using independent threads or processes which can be scheduled on multiple machines with little interprocess communication.

3. Pseudo and Quasi-Random Numbers

Pseudo-random numbers (PRNs) [10] are constructed to mimic truly random numbers. By virtue of Central Limit Theorem [11], the uncertainty in an average taken from \( N \) samples is \( O(N^{-1/2}) \). A generic approach to improving convergence is through the use of quasi-random numbers (QRNs) [4-6]. QRNs are deterministic sequences of variables that are particularly well distributed with correlations between points. With QRNs, the convergence of integration can sometimes be improved to \( O(N^{-1}) \).

The absolute error can be written in the form

\[ E = \frac{C}{N^r}, \]

where \( E \) is the absolute error, \( C \) is a constant, \( N \) is the number of integration samples and \( r \) is an exponent that depends on the kind of random number sequence used. Taking logarithm on the either side for Eq. (5) yields

\[ \ln(E) = \ln(C) - r \ln(N). \]

The exponent \( r \) is expected to be around \( 1/2 \) for PRNs and around 1 for QRNs. In a previous work [12], we have presented the results for the extraction of partial inductances through the use of PRNs. In this work, we extend that work to the use of QRNs. The results for three QRN sequences—Halton, Sobol and Niederreiter, are given in this paper.

4. Results

The algorithm has been benchmarked against several mutual inductance extraction problems in one, two and three dimensions. The numerical results have been matched with analytical solutions given in Ref. [13] and excellent agreement has been obtained for both PRNs and QRNs. The benchmark problem geometries are presented in Figures (1) to (5). In these benchmark problems, \( A = B = C = D = R = T = 5 \ \mu\text{m} \) and \( Q = S = U = V = 1 \ \mu\text{m} \). Table 1 includes the results for 10000 integration samples and it can be seen that the use of QRNs improves the results significantly. Figure (6) plots the least square fit of the logarithm of the absolute error from the analytical solution with respect to the logarithm of the number of integration samples for Benchmark Problem # 5 using Pseudo, Halton, Sobol and Niederreiter sequences. It can be seen that the exponent \( r \) follows the theoretically expected behavior.

The algorithm has been parallelized using message passage interface (MPI) and distributed on a cluster of 8 nodes, each of which contains a Dual Core AMD Opteron at 2 GHz frequency with 2 GB of RAM. The PRN (Mersenne Twister) and QRN source codes were based on the GNU Scientific Library implementation [14]. As shown in Figure (7) an almost linear speedup was achieved for both the PRN- and QRN-based implementations.
Table 1: Analytical and numerical results for the benchmarks with 10000 integration samples: A—Analytical, B-Pseudo, C-Halton, D-Sobol, E-Niederreiter. All dimensions are in pH.

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<th>Probs.</th>
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<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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</tr>
</tbody>
</table>

Fig. 1. Two parallel filaments of negligible width and thickness. The currents are in the $x$-direction.

Fig. 2. Two parallel tapes of negligible thickness. The currents are in the $z$-direction.
Fig. 3. Two tapes of negligible thickness, whose axis are parallel and widths are perpendicular. The currents are in the $z$-direction.

Fig. 4. A thin filament of negligible width and thickness is placed parallel to a rectangular bar. The currents are in the $z$-direction.

Fig. 5. Two rectangular bars placed parallel to each other. The currents are in the $z$-direction.
5. Conclusion and Future Work

Summarizing, a Monte Carlo algorithm for the extraction of partial inductances in IC interconnect structures has been developed through the use of QRNs. The algorithm has been validated with the help of analytical benchmarks. The methodology has been parallelized and an almost linear rate of parallelization is obtained. We believe that with additional development, this algorithm can be developed into an IC CAD tool for inductance extraction.

6. References


